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MONOTONICITY PRINCIPLE IN THE RAYLEIGH PROBLEM FOR AN ISOTHERMALLY INCOMPRESSIBLE FLUID

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The convection of an isothermally incompressible fluid in a horizontal layer with free undeformable boundaries kept at a constant temperature is considered. Under the fairly common assumptions of the temperature dependence of the specific volume, it is shown that the monotonicity principle holds and that the spectrum of critical Rayleigh numbers is countable and prime. Models with linear and quadratic temperature dependences of the specific volume are given as examples. The results on the spectrum of the critical Rayleigh numbers are also valid for some other boundary conditions.

Key words: *convection, isothermally incompressible fluid, monotonicity principle, Rayleigh number, oscillation operators.*

1. Formulation of the Problem. Let a viscous heat-conducting fluid fill an infinite horizontal layer of thickness H, bounded by free undeformable boundaries kept at constant temperatures: T_1 on the bottom wall and T_2 on the top wall ($T_2 < T_1$, i.e., the layer is heated from below). The fluid is acted upon by gravity with an acceleration $\boldsymbol{g}$.

We assume that the fluid is isothermally incompressible, i.e., its specific volume V depends only on the temperature. Let the function $V(T)$ be given by the formula

$$V = \tilde{V}(1 + \gamma F(T)), \qquad V(T) > 0, \tag{1.1}$$

where $\gamma > 0$ is a constant; $F(T)$ is a continuously differentiable function, $F(\tilde{T}) = 0$, $\tilde{V} = V(\tilde{T})$, and $\tilde{T} < T_1$ is a certain temperature value.

The heat-transfer equation can be considerably simplified by assuming that the viscosity η, thermal conductivity $\varkappa$, and specific heat at constant pressure c_p are constants and that the change in the potential energy of a fluid particle due to convection is small compared to the change in the internal energy. Under the above assumptions, the convection equations and the boundary conditions for the velocity $\boldsymbol{v}$, temperature T, and pressure p are written as [1]

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0, \qquad \rho = \frac{1}{V(T)},$$

$$\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\nabla\boldsymbol{v}\Big) = -\nabla p + \eta\Delta\boldsymbol{v} - \rho g\boldsymbol{k}, \qquad \rho\Big(\frac{\partial T}{\partial t} + \boldsymbol{v}\nabla T\Big) = \varkappa\Delta T,$$

$$T\Big|_{z=0} = T_1, \qquad T\Big|_{z=H} = T_2,$$

$$\frac{\partial v_1}{\partial z}\Big|_{z=0,H} = \frac{\partial v_2}{\partial z}\Big|_{z=0,H} = v_3\Big|_{z=0,H} = 0,$$

where $\boldsymbol{k} = (0, 0, 1)$ is the unit vector of the axis directed upward and $\boldsymbol{g} = -g\boldsymbol{k}$.

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Following [2], we make the variables dimensionless. As the length and temperature scales, we use the characteristic size h and the characteristic temperature difference $\Theta = T_1 - \tilde{T}$. The temperature of the lower boundary of the layer T_1 will be considered initial. As the time unit we use $\tau = \sqrt{h/(\alpha g \Theta)}$ is the characteristic time of convective rise of a heated fluid particle (or immersion of a cooled particle) (α is the average volume-expansion coefficient of the fluid). For the specific volume, velocity, and pressure we use the scales $\tilde{V}$, h/τ, and $gh\alpha\Theta/\tilde{V}$, respectively. Retaining the former notation for the dimensionless variables, we have

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0, \qquad \rho = V(T)^{-1}, \qquad V(T) = 1 + \beta\Phi(T),$$
$$\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\nabla\boldsymbol{v}\Big) = -\nabla p + \mu\Delta\boldsymbol{v} - \beta^{-1}(\rho - 1)\boldsymbol{k}, \qquad \rho\Big(\frac{\partial T}{\partial t} + \boldsymbol{v}\nabla T\Big) = \delta\Delta T; \tag{1.2}$$

$$T\Big|_{z=0} = 0, \qquad T\Big|_{z=l} = -l; \tag{1.3}$$

$$\frac{\partial v_1}{\partial z}\Big|_{z=0,l} = \frac{\partial v_2}{\partial z}\Big|_{z=0,l} = v_3\Big|_{z=0,l} = 0, \tag{1.4}$$

where $\beta = \alpha\Theta$, $\mu = \eta\tau\tilde{V}/h^2$, and $\delta = \varkappa\tau\tilde{V}/(h^2 c_p)$ are the dimensionless thermal-expansion coefficient, viscosity, and thermal conductivity, respectively [2], $l = H/h$ is the dimensionless thickness of the layer, and $\Phi(T) = F(T\Theta + T_1)/F(T_1)$ is a continuously differentiable function [$\Phi(-1) = 0$].

Next, we assume that $\beta \geq 0$ and consider only fluids expanding under heating (in the case of no density inversion).

Problem (1.2)–(1.4) has a steady-state solution which corresponds to the mechanical equilibrium of the fluid:

$$\boldsymbol{v}_0 = 0, \qquad T_0 = -z, \qquad p_0 = \beta^{-1}\int_z^l [\rho(-s) - 1]\, ds + \text{const.} \tag{1.5}$$

Because the equilibrium temperature profile is linear, it follows that $l = (T_1 - T_2)/\Theta$.

Since $\Phi(T)$ is a continuously differentiable function, the dependence $V(T)$ in the vicinity of the point $\tilde{T}$ can be written as

$$V(\tilde{T} + T) = 1 + \beta\Phi(\tilde{T}) + \beta\frac{\partial\Phi}{\partial T}\Big|_{T=\tilde{T}} T + \beta O(T^2), \qquad T \to 0. \tag{1.6}$$

The secondary solutions p', $\boldsymbol{v}'$, and T' of problem (1.2)–(1.4) will be sought in the form

$$p' = p_0 + \mu\delta p, \qquad \boldsymbol{v}' = \boldsymbol{v}_0 + \delta\boldsymbol{v}, \qquad T' = T_0 + T. \tag{1.7}$$

Substituting (1.7) into (1.2)–(1.4) and using (1.5) and (1.6), we obtain the following nonlinear system for perturbations of p, $\boldsymbol{v}$, and T (the primes are omitted):

$$V(T - z) = 1 + \beta\Phi(T - z),$$
$$\beta\frac{\partial\Phi}{\partial T}\Big|_{T=-z}\Big(\frac{\partial T}{\partial t} - v_3 + \boldsymbol{v}\nabla T\Big) = V(T - z)\operatorname{div}\boldsymbol{v},$$
$$\mathrm{Pr}^{-1}\rho(T - z)\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\nabla\boldsymbol{v}\Big) = -\nabla p + \Delta\boldsymbol{v} + \mathrm{R}\rho_0\rho(T - z)\Big(\frac{\partial\Phi}{\partial T}\Big|_{T=-z} T + O(T^2)\Big)\boldsymbol{k}, \tag{1.8}$$
$$\rho(T - z)\Big(\frac{\partial T}{\partial t} - v_3 + \boldsymbol{v}\nabla T\Big) = \Delta T.$$

Here $\rho_0 = \rho_0(z) = \rho(-z)$ is the density in the equilibrium state (1.5) and $\mathrm{R} = (\mu\delta)^{-1}$ and $\mathrm{Pr} = \mu\delta^{-1}$ are the Rayleigh and Prandtl numbers, respectively [2]. Because the function of the specific volume (1.1) is positive, the equilibrium density $\rho_0(z) = V(-z)^{-1}$ is positive in the interval $z \in [0, l]$.

From (1.3) and (1.4), we obtain the following boundary conditions for the temperature and velocity perturbations:

$$T\Big|_{z=0} = 0, \qquad T\Big|_{z=l} = 0; \tag{1.9}$$

$$\frac{\partial v_1}{\partial z}\Big|_{z=0,l} = \frac{\partial v_2}{\partial z}\Big|_{z=0,l} = 0, \qquad v_3\Big|_{z=0,l} = 0. \tag{1.10}$$

The linearized system for the perturbations that corresponds to (1.8) is written as

$$\beta\rho_0(z)\phi(z)\Big(\frac{\partial T}{\partial t} - v_3\Big) = \operatorname{div}\boldsymbol{v},$$

$$\mathrm{Pr}^{-1}\rho_0(z)\,\frac{\partial \boldsymbol{v}}{\partial t} = \Delta\boldsymbol{v} - \nabla p + \mathrm{R}\rho_0^2(z)\phi(z)T\boldsymbol{k}, \tag{1.11}$$

$$\rho_0(z)\Big(\frac{\partial T}{\partial t} - v_3\Big) = \Delta T,$$

where $\phi(z) = (\partial\Phi/\partial T)\Big|_{T=-z}$ is a continuous function.

2. Spectral Problem. Monotonicity Principle. The nontrivial solutions of the boundary-value problem (1.11), (1.9), (1.10) that are periodic in the variable x_1 with period $2\pi/k_1$ and in the variable x_2 with period $2\pi/k_2$ are sough in the form

$$\boldsymbol{v}(x_1, x_2, z) = \boldsymbol{U}(z)\exp(-\sigma t + ik_1x_1 + ik_2x_2),$$

$$T(x_1, x_2, z) = \theta(z)\exp(-\sigma t + ik_1x_1 + ik_2x_2), \qquad p(x_1, x_2, z) = p(z)\exp(-\sigma t + ik_1x_1 + ik_2x_2),$$

where k_1 and k_2 are wavenumbers and σ is the perturbation decrement, which generally can be complex. It is assumed that the average fluid mass flow in the directions x_1 and x_2 is absent:

$$\int\limits_{-\pi/k_2}^{\pi/k_2}\int\limits_0^l \rho v_1\,dx_2\,dx_3 = \int\limits_{-\pi/k_1}^{\pi/k_1}\int\limits_0^l \rho v_2\,dx_1\,dx_3 = 0.$$

Separating the variables, we obtain the following spectral problem for ordinary differential equations:

$$f_0Dw + F = -\sigma\beta\rho_0\phi\theta; \tag{2.1}$$

$$LF + k^2p = -\sigma\mathrm{Pr}^{-1}\rho_0F; \tag{2.2}$$

$$Lw - Dp + \mathrm{R}\phi\rho_0^2\theta = -\sigma\mathrm{Pr}^{-1}w; \tag{2.3}$$

$$L\theta + w = -\sigma\rho_0\theta; \tag{2.4}$$

$$z = 0, l\colon \qquad \theta = w = DF = 0. \tag{2.5}$$

Here $\rho_0(z) = \rho(-z)$, $f_0(z) = \rho_0(z)^{-1}$, $w = \rho_0U_3$, $F = ik_1U_1 + ik_2U_2$, $D = d/dz$, $k^2 = k_1^2 + k_2^2$, and $L = D^2 - k^2$.

Let us show that the monotonicity principle holds in this case, i.e., that the condition $\operatorname{Re}\sigma = 0$ implies $\operatorname{Im}\sigma = 0$. Eliminating the functions F, p, and w from system (2.1)–(2.5), for the unknown function θ we obtain the following boundary-value eigenvalue problem for the perturbation decrements σ:

$$-LNL\theta - \sigma\Big(\frac{1}{\mathrm{Pr}} + 1\Big)L^2\theta - \frac{\sigma^2}{\mathrm{Pr}}\,M\theta = \mathrm{R}k^2\phi\rho_0^2\theta; \tag{2.6}$$

$$z = 0, l\colon \qquad \theta = L\theta = NL\theta = 0. \tag{2.7}$$

Here the differential expressions for N and M are written as

$$N = D[f_0D] - k^2f_0, \qquad M = D[\rho_0D] - k^2\rho_0. \tag{2.8}$$

Along with the solution θ of system (2.6), (2.7), we examine the complex conjugate solution $\overline{\theta}$. Multiplying Eq. (2.6) into $\overline{\theta}$ and integrating it by parts with respect to z from 0 to l, we obtain

$$\sigma^2I_1 - \sigma I_2 + I_3 = \mathrm{R}I_4, \tag{2.9}$$

where the quantities

$$I_1 = \frac{1}{\mathrm{Pr}}\Big(\int\limits_0^l \rho_0 |D\theta|^2 dz + k^2 \int\limits_0^l \rho_0 |\theta|^2 dz\Big),$$

$$I_2 = \Big(\frac{1}{\mathrm{Pr}} + 1\Big)\int\limits_0^l |L\theta|^2 dz, \qquad I_3 = \int\limits_0^l f_0 |DL\theta|^2 dz$$

are positive because $\theta(z)$ is a nontrivial solution of problem (2.6), (2.7), and the quantity $I_4 = k^2 \int\limits_0^l \phi\rho_0^2 |\theta|^2 dz$ is real. Dividing the imaginary and real parts in Eq. 2.9), we have

$$\operatorname{Im}\sigma(2\operatorname{Re}\sigma I_1 - I_2) = 0. \tag{2.10}$$

According to (2.10), $\operatorname{Im}\sigma = 0$ or $\operatorname{Re}\sigma = I_2/(2I_1) > 0$. Hence, oscillatory modes can occur only in the case of stability ($\operatorname{Re}\sigma > 0$). Thus, the monotonicity principle is proved.

We note that the above proof takes into account the cases of negative critical Rayleigh numbers, such as, for example, those in the penetrative convection model [3–5].

Remark 1. In the case of different boundary conditions, we were unable to prove the absence of oscillatory instability. Numerical calculations [6] show that, in the case of two solid walls, oscillatory stability occurs under heating from both below and from above, while oscillatory instability has not been found numerically [4, 6].

3. Spectrum of Critical Rayleigh Numbers. Setting $\sigma = 0$ in (2.6) and (2.7), we consider the system of equations for neutral perturbations:

$$-LNL\theta = \mathrm{R}k^2\phi\rho_0^2\theta; \tag{3.1}$$

$$z = 0, l: \qquad \theta = L\theta = NL\theta = 0. \tag{3.2}$$

Problem (3.1), (3.2) is an eigenvalue problem, in which the critical Rayleigh numbers play the role of eigenvalues.

We study fluids of two types: normal and abnormal. Fluids expanding monotonically under heating will be called normal fluids. In this case, $\partial F/\partial T > 0$ for all $T \in [T_2, T_1]$ and $\phi(z) > 0$ for all $z \in [0, l]$ (here it is assumed that $h = H$, $\tilde{T} = T_2$, and $\tilde{V} = V(T_2)$, and hence $l = 1$). This class of fluids includes, for example, fluids with a linear temperature dependence of the specific volume $V = \tilde{V}(1 + \alpha(T - T_2))$.

Fluids with density inversion will be called abnormal fluids. A fluid is considered normal if there are no density inversion points in the interval (T_2, T_1) and the fluid expands under heating. At the density inversion points, the function $\partial F/\partial T$ changes sign. We will further consider only the case where in the interval (T_2, T_1) this function changes sign once at the point T_*. Then, the function $\phi(z)$ also changes sign once at the point $z = 1$ [here we set $h = H\Theta/(T_1 - T_2)$, $\tilde{T} = T_*$, and $\tilde{V} = V(T_*)$, and hence $l = H/h > 1$]. As an example of such fluids is water at atmospheric pressure (see, e.g., [3–5]), for which the inversion temperature $T_* \approx 4°\mathrm{C}$ and $V = \tilde{V}(1 + \gamma(T - T_*)^2)$ at $T_1 > 4°\mathrm{C}$ and $T_2 < 4°\mathrm{C}$.

Remark 2. In the case of abnormal fluids, it is of no insignificance whether the layer is heated from below or from above; therefore, the further results obtained for fluids with specific volume inversion are also valid for $T_2 > T_1$ (in this case, negative critical numbers Rayleigh appear).

Let $\beta_* > 0$ be the first number such that $\rho_0(z_0, \beta_*) = 0$ for a certain z_0 in the interval $[0, l]$ [if $\rho_0(z, \beta_*) > 0$ for any $z \in [0, l]$ and $\beta > 0$, we set $\beta_* = \infty$]. Then, the following theorem holds.

Theorem 1. *Let* $\beta \in [0, \beta_*)$, $k \geq 0$. *Then, for a normal fluid* [$\phi(z) > 0$]*, the spectrum of problem* (3.1), (3.2) *consists of a countable number of positive prime eigenvalues*

$$0 < \mathrm{R}_1 < \mathrm{R}_2 < \mathrm{R}_3 < \ldots\ ,$$

and for an abnormal fluid, [$\phi(z)$ *changes sign once*]*, it consists of a countable number of positive and negative prime eigenvalues*

$$\ldots < \mathrm{R}_{-3} < \mathrm{R}_{-2} < \mathrm{R}_{-1} < 0 < \mathrm{R}_1 < \mathrm{R}_2 < \mathrm{R}_3 < \ldots\ .$$

To prove the theorem, we use the following lemma [7].

Lemma 1. *We consider the differential expression of the second order*

$$M_2 = \frac{d}{dz}\Big[f_1(z)\,\frac{d}{dz}\Big] - f_2(z).$$

Let $f_1(z)f_2(z) > 0\ \forall z \in [a,b]$. Then, M_2 can be factorized, i.e., represented as

$$M_2 = \frac{1}{y}\,\frac{d}{dz}\,f_1 y^2\,\frac{d}{dz}\,\frac{1}{y},$$

where $y(z)$ is the solution of the equation $M_2 y = 0$ that does not have zeros in the interval $[a,b]$.

The differential expression of N from (2.8) defined in the interval $[0,l]$ satisfies the conditions of Lemma 1 and, hence, can be factorized with positive weights:

$$N = \rho_1\,\frac{d}{dz}\,\rho_2\,\frac{d}{dz}\,\rho_3 \tag{3.3}$$

($\rho_1 = \rho_3 = 1/u$, $\rho_2 = f_0 u^2$, and u is the solution of the equation $Nu = 0$ that does not have zeros on the interval $[0,l]$).

Lemma 1 also implies the well-known factorization

$$L = \mathrm{e}^{-kz}\,\frac{d}{dz}\,\mathrm{e}^{2z}\,\frac{d}{dz}\,\mathrm{e}^{-kz}\,. \tag{3.4}$$

Next, we use the following theorems.

Theorem 2 (Kalafati–Gantmacher–Krein [8, 9])**.** *In the spectral problem*

$$L_1 y = l_0 y^{(n)} + l_1 y^{(n-1)} + \ldots + l_n y = \lambda r y,$$

$$U_i y = y^{(q_i)}(a) + \sum_{q<q_i} \gamma_{iq} y^{(q)}(a) = 0, \qquad i = 1,\ldots,m, \tag{3.5}$$

$$U_i y = y^{(q_i)}(b) + \sum_{q<q_i} \gamma_{iq} y^{(q)}(b) = 0, \qquad i = m+1,\ldots,n,$$

where γ_{iq} are real, $l_0(x), l_1(x),\ldots,l_n(x)$ are smooth real functions in $[a,b]$, $l_0(x) \neq 0$, $r(x)$ is a continuous function positive in $[a,b]$, $1 \le m \le n-1$, $0 \le q_1 < \ldots < q_m \le n-1$, and $0 \le q_{m+1} < \ldots < q_n \le n-1$, let the differential expression of $L_1 y$ admit the factorization

$$L_1 y = r_0\,\frac{d}{dz}\,r_1\,\frac{d}{dz}\,\cdots\,r_{n-1}\,\frac{d}{dz}\,r_n y,$$

where $(-1)^{n-m} r_0(x)\cdots r_n(x) > 0$ $(a \le x \le b)$, and let the boundary conditions be nonsingular and representable as

$$\begin{gathered}\sum_{q=1}^{n} \alpha_{iq}(D_{q-1}y)(a) = 0, \quad i = 1,2,\ldots,m,\\ \sum_{q=1}^{n} \beta_{iq}(D_{q-1}y)(b) = 0, \quad i = 1,2,\ldots,n-m;\end{gathered} \tag{3.6}$$

$$D_0 u = r_n u, \qquad D_m u = r_{n-m}\,\frac{d}{dz}\,[D_{m-1}u]. \tag{3.7}$$

If all nonzero minors of order m of the matrix

$$A = \|(-1)^q \alpha_{iq}\| \qquad (i = 1,\ldots,m;\quad q = 1,\ldots,n)$$

have identical signs and the same is valid for the minors of order $(n-m)$ of the matrix

$$B = \|\beta_{iq}\| \qquad (i = 1,\ldots,n-m;\quad q = 1,\ldots,n),$$

the boundary-value problem (3.5) *has oscillatory Green's function $G(x,t)$ and, hence, a prime positive real spectrum*

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots\ .$$

Theorem 3 (of Barkovskii–Yudovich[10]). *In the conditions of Theorem 2, if the function $r(x)$ in $[a, b]$ changes sign once, the boundary-value problem* (3.5) *has a countable number of prime negative and prime positive eigenvalues*

$$\ldots < \lambda_{-3} < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots .$$

To prove Theorem 1, we write (3.1) and (3.2) as

$$-f_0^2 LNL\theta = \mathrm{R}k^2\phi\theta; \tag{3.8}$$

$$z = 0, l: \qquad \theta = L\theta = NL\theta = 0. \tag{3.9}$$

From (3.3) and (3.4) it follows that the differential expression $f_0^2 LNL$ can be factorized with positive weights:

$$f_0^2 LNL = r_0 \frac{d}{dz} r_1 \frac{d}{dz} r_2 \frac{d}{dz} r_3 \frac{d}{dz} r_4 \frac{d}{dz} r_5 \frac{d}{dz} r_6, \tag{3.10}$$

which are given by the equalities $r_0 = f_0^2\, \mathrm{e}^{-kz}$, $r_1 = r_5 = \mathrm{e}^{2kz}$, $r_2 = \mathrm{e}^{-kz}\,\rho_1$, $r_3 = \rho_2$, $r_4 = \rho_3\,\mathrm{e}^{-kz}$, and $r_6 = \mathrm{e}^{-kz}$.

Using the notation (3.7), in which $n = 6$ and $m = 3$, we reduce problem (3.8), (3.9) to the form

$$-D_6\theta = \mathrm{R}k^2\phi\theta;$$

$$z = 0, l: \qquad D_0\theta = D_2\theta = D_4\theta = 0. \tag{3.11}$$

Boundary conditions (3.11) can be written in the form of (3.6), where the matrices $A = (\alpha_{iq})$ and $B = (\beta_{jq})$ have the form

$$A = B = \begin{pmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \end{pmatrix}.$$

It is obvious that all nonzero minors of the 3rd order of these matrices are positive. In the case where $\phi(z) > 0$ in the interval $z \in [0, l]$ for any $\beta \in [0, \beta_*)$ and $k \geq 0$, the boundary-value problem (3.8), (3.9), in view of (3.10), satisfies the conditions of Theorem 2, which implies the statement of the first part of Theorem 1. If the function $\phi(z)$ changes sign once in the interval $z \in [0, l]$ for any $\beta \in [0, \beta_*)$ and $k \geq 0$, the positiveness of all minors of the 3rd order of the matrices A and B and the factorization (3.10) allow Theorem 3 to be applied to the spectral problem (3.8), (3.9). Theorem 3 implies the statement of the second part of Theorem 1. Theorem 1 is proved.

Remark 3. If $\phi(z) > 0$, the boundary-value problem (3.8), (3.9) has oscillatory Green's function under Theorem 2. According to the theory of integral operators with oscillatory kernels [9], the eigenvector-function of this problem (θ_1, w_1) that corresponds to the minimum eigenvalue R_1 does not change sign in $[0, l]$, and the eigenector-function (θ_n, w_n) corresponding to the nth (in modulus) eigenvalue R_n changes sign $n - 1$ times in $[0, l]$.

Remark 4. Theorem 1 is valid, for example, for the cases where the boundaries of the layer are solid walls or where one of them is a solid wall and the other is a free undeformable boundary. Then, on the boundary instead of the condition $DF = 0$ [see (2.5)], the condition $F = 0$ is valid and the spectral problem changes accordingly: instead of $NL\theta = 0$, we have $DL\theta = 0$.

Let us consider some examples.

3.1. *Linear Temperature Dependence of Specific Volume.* In his case, the function $V(T)$ is written as

$$V = \tilde{V}(1 + \alpha(T - \tilde{T})). \tag{3.12}$$

As shown in [11], for this (and only this) temperature dependence of the specific volume, the specific heat at constant pressure c_p does not depend on pressure (see also [2, 4, 6, 12–14]).

Since the dependence $\partial V/\partial T = \tilde{V}\alpha$ does not have zeros, we can set $\tilde{T} = T_2$. Then, $l = 1$ and $\phi(z) \equiv 1$, and, under Theorem 1, the spectrum of the critical Rayleigh numbers consists of a countable numbers of prime positive eigenvalues (critical Rayleigh numbers).

Remark 5. The statements of Theorem 1 remain valid if $\beta \to 0$ in (1.11). If the equation of state has the form (3.12), then, for $\beta \to 0$ ($\alpha \to 0$), Eq. (1.11) lead to Oberbeck–Boussinesq approximation equations [2, 4, 6, 14]. Then, Theorem 1 implies that the spectrum of critical Rayleigh numbers is positive and prime for the Oberbeck–Boussinesq approximation [15].

3.2. *Quadratic Temperature Dependence of Specific Volume.* We consider a penetrative convection model with a quadratic temperature dependence of the specific volume [3–5]:

$$V = V_i(1 + \gamma(T - T_*)^2).$$

Here V_i is the minimum specific volume, T_* is the inversion point, γ is a constant, $\alpha = \gamma\Theta$ is the average volume-expansion coefficient of the fluid, $\beta = \alpha\Theta = \gamma\Theta^2$ is the thermal expansion parameter, and the derivative $\partial V/\partial T = 2V_i\gamma(T - T_*)$ changes sign at the point T_*. Then, $l > 1$ and, in the interval $\phi(z) = 2(1 - z)$, the function $z \in [0, l]$ changes sign once at the point $z = 1$. According to Theorem 1, in this case, a countable number of positive and negative prime eigenvalues (critical Rayleigh numbers) exists and imaginary and multiple eigenvalues are absent.

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